Metastable states in disordered Ising magnets in mean-field approximation

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Abstract

The mechanism of appearance of exponentially large number of metastable states in magnetic phases of disordered Ising magnets with short-range random exchange is suggested. It is based on the assumption that transitions into inhomogeneous magnetic phases results from the condensation of macroscopically large number of sparse delocalized modes near the localization threshold. The properties of metastable states in random magnets with zero ground state magnetization (dilute antiferromagnet, binary spin glass, dilute ferromagnet with dipole interaction) has been obtained in framework of this mechanism using variant of mean-field approximation. The relations between the characteristics of slow nonequilibrium processes in magnetic phases (such as hysteresis loop form, thermo-remanent and isothermal remanent magnetizations, field-cooled and zero-field-cooled thermodynamic quantities) and thermodynamic parameters of metastable states are established.

1 Introduction

The specific feature of disordered magnets is the appearance of large number of metastable states in magnetic phases. They cause various irreversibility phenomena in the reaction to the changes of temperature and external magnetic field, such as the dependence of thermodynamic parameters on the order and the rate of these changes, the appearance of hysteresis loops, depending on the field amplitude etc. [1, 2, 3, 4]. These effects appear to be common to the all types of magnetic disorder - from dilute magnets with nonmagnetic impurities [2, 4] to spin glasses existing in solid solutions of ferromagnets with antiferromagnets [1, 3]. This universality of nonequilibrium phenomena compels to suggest the existence of common mechanism responsible for the appearance of metastable states in random magnets.

The evidences of the existence of metastable states, the number of which growth exponentially with the number of sites, has been obtained in numerical studies of several models of disordered magnets with short-range exchange [5, 6, 7, 8]. Theoretical description of the related inergodic phenomena is also possible mainly with the use of numerical methods, see e. g. [1, 4, 9].

The principal analytical results has been obtained in the Sherrington-Kirkpatrick spin glass model with infinite-range interaction. It was established that in this model the region of inergodicity exists in the external field less than the Almeida-Thouless field H_{AT}, [10], and in the framework of the replica symmetry breaking scheme by Parisi [11] and the notion of the hierarchy of macroscopic relaxation times [12] the thermodynamic parameters of field-cooled and zero-field-cooled processes has been obtained [1]. But this is all what is known about the irreversible processes in this model. In spite of the wide application of such approach to other mean-field models, see e.g. [13, 14], it is not clear how all variety of inergodic phenomena resulting from transitions between metastable states could be described by the methods of [11, 12].

Meanwhile, theoretical description of irreversible processes could be made quite simple and apparent if thermodynamic properties of metastable states, their regions of stability and points of possible transitions between them would be known at $H < H_{AT}$. Principally, in the mean-field models this data could be obtained from (nonaveraged)equations for local magnetic moments. The most popular example of application of this method in the theory of disordered magnets is the TAP-equations for local magnetic moments in the Sherrington-Kirkpatrick spin glass model [15]. It was established that the number of their solutions is exponentially large [16], yet their explicit form has not been determined by the analytical

methods. Now it is still not clear how many of these solutions correspond to the stable states and how the divergent barriers between them emerge.

Meanwhile, the results of the investigation of TAP-equations in case of large but finite interaction range [17] cast some doubt on the possibility to use the infinite-range models as a valid approximation for description of real disordered magnets. The authors of [17] first noticed the consequences of the qualitative difference of the spectrum of the gaussian orthogonal ensemble of random exchange in Sherrington-Kirkpatrick model with all eigenvectors delocalized and that of short-range random exchange having localized eigenvectors near its edges [18]. It was shown in [17] that in $3d$ spin glass with sufficiently large but finite interaction range, the condensation of localized modes could drastically change the nature of phase transition preventing the subsequent condensation of delocalized modes if localization length index is greater than 2/3. This case with the absence of metastable states seems to be not very common in real disordered magnets but this result demonstrates the necessity to take into account the qualitative features of spectral characteristics of short-range random exchange.

The studies of the spectral structure of various short-range random matrix ensembles have shown that all they have similar features having localized eigenvectors near the edges and fractal structure of eigenvectors at the localization threshold [18, 19, 20]. Thus one can suppose that universality of the inergodic phenomena in the different types of disordered magnets results from the similarity of their spectral properties. The common mechanism of the appearance of large number of metastable states in such systems could be an almost simultaneous condensation of macroscopically large number of almost nonoverlapping fractal modes near the localization threshold.

Here we must note that the finiteness of the interaction range does not forbid the quantitative description of such mechanism in the framework of the mean-field approach. Indeed, the macroscopic number of condensing modes (order parameter components) greatly reduce the number of noncondensing ones (order parameter fluctuations). So we may expect that Ginzburg parameter will be essentially smaller than that of the homogeneous magnet of the same dimension for all types of short-range random exchange.

On the basis of such suppositions, we attempt here to develop the phenomenological mean-field theory of disordered Ising magnets with the zero ground state magnetization, using some heuristic suppositions about the structure of the fractal eigenvectors in random exchange ensembles. In this approach the appearance of numerous metastable states could be naturally explained and, using some simplifying assumptions, it is possible to describe analytically the thermodynamic properties of these states near transition point. In their turn, these results allow to determine parameters of all slow irreversible processes and, particularly, to get the analytical expressions, describing the hysteresis loop form at arbitrary amplitude of external AC field as well as field and temperature dependencies of remanent magnetizations in the considered types of disordered magnets.

2   Mean-field approximation for disordered Ising magnets

The Hamiltonian of disordered Ising magnet is

$$H = -\frac{1}{2}\sum_{ij} J_{ij} S_i S_j \tag{1}$$

Here J_{ij} is the random exchange matrix, $S_i = \pm 1$. In the majority of realistic models, J_{ij} are nonzero only for the sites within several coordination spheres and

$$|J_{ij}| < J_{\max}$$

Here we consider just such models.

The mean-field approximation for the Hamiltonian in Eq. 1 consists in the substitution of matrix J_{ij} by the projector on its largest eigenvalue

$$J_{ij} \approx J \sum_{\alpha=1}^{N_0} c_i^\alpha c_j^\alpha \tag{2}$$

Here c_i^α are the normalized eigenvectors of J_{ij} corresponding to the largest (degenerate in case of multi-sublattice antiferromagnet) eigenvalue J, $\alpha = 1, ..., N_0$.

Then the Hamiltonian becomes the function of (multicomponent) order parameter

$$\eta^\alpha = N^{-1/2} \sum_i^N c_i^\alpha S_i$$

and one should only calculate the entropy

$$S\left(\eta^{\alpha}\right) = \ln Tr\delta_{N^{1/2}\eta^{\alpha},\sum c_i^{\alpha}S_i} \tag{3}$$

to get the inequilibrium thermodynamic potential

$$F\left(\eta^{\alpha}\right) = -\frac{NJ}{2}\sum_{\alpha=1}^{N_0}\left(\eta^{\alpha}\right)^2 - TS\left(\eta^{\alpha}\right)$$

Minimization of $F\left(\eta^{\alpha}\right)$ would give the equilibrium values of thermodynamic parameters, corresponding to the lowest minimum, and those of metastable states, corresponding to less deep minima. In particular, one can get the average spin values

$$\langle S_i\rangle_T = N^{1/2}\sum_{\alpha=1}^{N_0} c_i^{\alpha}\eta^{\alpha} \tag{4}$$

In some cases this approximation may give quantitative description of thermodynamics of the second order transitions in homogeneous magnets (except for the fluctuation region in the immediate vicinity of transition point). So it is rather natural to use the mean-field approximation for the disordered magnets. In this case it seems sufficient to average the results obtained in the framework of the described above scheme over random J and c_i^{α}. Then the thermodynamics of disordered magnets would not qualitatively differ from that of ideal magnets and, in particular, there would not appear numerous metastable states in the inhomogeneous magnetic phases.

The most probable reason for the emergence of exponentially large number of metastable states lies in the specific structure of spectrum and eigenvectors of random matrix J_{ij}. Indeed, the eigenvectors of J_{ij} having the described above properties are localized near the upper (and lower) boundary of the spectrum [18]. So to describe the transition in random magnet one should not take in the Eq. 2 the largest eigenvalue but rather lower one J at the localization threshold, i. e. the largest eigenvalue from those having delocalized eigenvectors. The reason is that the macroscopic transition could take place only as a result of condensation of delocalized mode, while the preceding condensation of local modes with larger eigenvalues (transition temperatures) results in the specific transition into Griffiths' phase, which is not accompanied by noticeable anomalies of thermodynamic parameters [21, 22]. Yet we must note that according to [17] in case of sufficiently large interaction range the condensation of local modes could stabilize the delocalized modes and macroscopic transition would not take place. So here we assume the range of interaction to be sufficiently small for macroscopic transition to occur.

Meanwhile, it seems rather probable that in the majority of random realizations the condensation of just one delocalized mode is not sufficient for the stabilization of new magnetic phase. For the dimension $d > 2$, delocalized eigenvectors near the localization threshold has rather sparse (fractal) structure consisting of rare localization regions connected just by the branching chains. In other words, there is the set of sites with the structure of percolation cluster [12], where $N(c_i^{\alpha})^2 >> 1$, while on the other sites $N(c_i^{\alpha})^2 << 1$. In this respect the modes which are close to the localization threshold differ essentially from those in the interior of the spectrum and from the modes of translationally invariant J_{ij} which have $N(c_i^{\alpha})^2 \approx 1$ at almost all sites.

The evidences of such fractal structure of these nearly localized modes were obtained in the numerical studies of various ensembles of short-range random matrices, see [19, 20] and references therein. According to Eq. 4 the condensation of one such mode results in the appearance of sufficiently large average spins only on a sparse fractal structure, which would not suffice to stabilize the modes with lower eigenvalues, being localized, in general, on the other fractal sets of sites. To be more precise, the condensation of the nearest to the localization threshold mode c_i^0 can stabilize only those modes with $J_\alpha < J$, which overlap essentially with it, i. e. having $Nc_i^{\alpha}c_i^0 >> 1$ at almost all sites where $N(c_i^{\alpha})^2 >> 1$. So after the condensation of the first sparse mode, the second mode having almost zero overlap with the first one will condense at lower temperature. Further decreasing of temperature will result in the condensation of third sparse mode which does not essentially overlap with the first and second ones and so on. This subsequent condensation of almost nonoverlapping modes with lower eigenvalues will take place until sufficiently large average spins appear at almost all sites. In the intervals between the eigenvalues of condensing modes there can exist, in general, an arbitrary numbers of modes which do not condense due to the large overlap with the previously condensed ones. These modes represents the order parameter fluctuations and should be omitted in the mean-field approximation.

Fractal structure of condensing modes suggests that their number diverges when $N \to \infty$. Indeed, if the sets of sites, where the modes considered are mainly localized, have the fractal dimension $d_f < d$, then the number of the condensing modes, N_0, is of the order $N^{1-d_f/d}$.

The described mechanism of the transition into inhomogeneous magnetic phases can rather naturally explain the appearance of exponentially large number of metastable states. Indeed, the condensation of one mode in zero field gives rise to two stable states related by the global spin reversal, while each subsequent condensation multiplies this number by factor two. Thus there appears $2^{N_0} \sim \exp(N^{1-d_f/d} \ln 2)$ stable states. Spin configurations in these states will be related by the overturns of independent groups of spins, corresponding to fractal modes. Just this structure of the set of ground states has been revealed in recent numerical study of $3d$ Ising spin glass with $\pm J$ exchange [13], which makes this mechanism rather probable.

The fact that overlaps of condensing modes almost vanish (i. e. $Nc_i^\alpha c_i^\beta << 1$ for $\alpha \neq \beta$ for almost all sites) allows to simplify essentially further considerations. Let us approximate the corresponding eigenvectors c_i^α by the set of nonoverlapping (normalized) vectors e_i^α, $e_i^\alpha e_i^\beta = 0$ for $\alpha \neq \beta$, which are equal to c_i^α in the regions where they are mainly localized ($N(c_i^\alpha)^2 >> 1$) and zero outside them. Then on the subspace, spread by the (apparently, orthogonal) vectors e_i^α, J_{ij} is almost diagonal

$$J_{ij} = \sum_{\alpha,\beta=1}^{N_0} (J\delta_{\alpha\beta} - J_{\alpha\beta}) e_i^\alpha e_j^\beta$$

Here $J_{\alpha\beta}$ is small positively defined matrix,

$$|J_{\alpha\beta}| << J$$

Then it is easy to find the mean-field thermodynamic potential, depending on the multicomponent order parameter

$$l_\alpha = N_\alpha^{-1/2} \sum_i e_i^\alpha S_i$$

$$N_\alpha = \sum_i \theta \left(|e_i^\alpha|\right)$$

($\theta(x)$ is Haviside's step function) and (quasi)local magnetizations

$$m_\alpha = N_\alpha^{-1} \sum_i S_i \theta \left(|e_i^\alpha|\right)$$

It has the form

$$F = -\frac{1}{2} \sum_{\alpha,\beta=1}^{N_0} \sqrt{N_\alpha N_\beta} \left(J\delta_{\alpha\beta} - J_{\alpha\beta}\right) l_\alpha l_\beta - T \sum_{\alpha=1}^{N_0} N_\alpha S_\alpha \left(l_\alpha, m_\alpha\right) - NHm \qquad (5)$$

$$S_\alpha \left(l_\alpha, m_\alpha\right) = N_\alpha^{-1} \ln Tr_\alpha \delta_{N_\alpha^{1/2} l_\alpha, \sum e_i^\alpha S_i} \delta_{N_\alpha m_\alpha, \sum S_i \theta(|e_i^\alpha|)} \qquad (6)$$

Here Tr_α denotes the sum over spin configuration of those sites where $e_i^\alpha \neq 0$. According to the above considerations, $N_\alpha \to \infty$ in thermodynamic limit,

$$\sum_{\alpha=1}^{N_0} N_\alpha = N$$

and homogeneous magnetization is

$$m = \sum_{\alpha=1}^{N_0} \frac{N_\alpha}{N} m_\alpha$$

Thermodynamic potential F, Eq. 5, depends on a small random matrix $J_{\alpha\beta}$ and random vectors e_i^α. Their form is determined by the type of the random exchange matrix ensemble. In some cases it is possible to get some notion on the e_i^α form. For example, in the spin glass with binary random exchange,

$$J_{ij} = \pm J_{\max}$$

in every bond configuration there are nonfrustrated d-dimensional clusters, i. e. the clusters which have unique spin configuration σ_i providing the energy minimum, and

$$J_{ij}\sigma_j \approx 2dJ_{\max}\sigma_i$$

Thus the delocalized eigenvectors with largest eigenvalues could be approximately constructed via connection of some nonfrustrated d-dimensional clusters by the branching chains without loops which are also nonfrustrated at all bond configurations [24]. So e_i^α can be approximately represented as

$$e_i^\alpha = N_\alpha^{-1/2} \sigma_i^\alpha \tag{7}$$

where σ_i^α are the spin configuration constructed as described on the nonfrustrated fractal sets of sites. In dilute magnets with the concentration of magnetic atoms above the percolation threshold, the vectors e_i^α can be also represented in the form Eq. 7 as one can connect by chains the d-dimensional ferromagnetic (antiferromagnetic) clusters belonging to the percolation cluster. So $\sigma_i^\alpha = 1$ in dilute ferromagnet and $\sigma_i^\alpha = (-1)^{\mathbf{kr}_i}$ in dilute antiferromagnet.

Yet we must note that in some specific random realization of matrix J_{ij} its eigenvectors near the localization threshold and e_i^α could essentially differ from that of Eq. 7. Nevertheless we will suppose that this expression approximates e_i^α reasonably well in the majority of random realizations so it can be used for the estimates of the ensemble average of the sums

$$u_{n\alpha} = N_\alpha^{n/2-1} \sum_i (e_i^\alpha)^n \tag{8}$$

As we show below, thermodynamics near the transition point does not depend on the detailed form of e_i^α being determined by the several constants $u_{n\alpha}$, Eq. 8.

The advantage of F representations in the form of Eq. 5 is the additivity of the entropy. But with arbitrary random matrix $J_{\alpha\beta}$ this expression is still difficult to analyze. We can simplify it using rather apparent consideration that eigenvalues of this matrix, lying in an interval between zero and some $J_0 << J$, must condense near zero as the farther from localization threshold the more rare are the modes which do not overlap with the preceding ones. Then, as the average interval between eigenvalues is of the order $1/N$, the same order could have the smallest eigenvalues of $J_{\alpha\beta}$. So we can approximate $J_{\alpha\beta}$ by the projector on some (random) vector r_α which properties are determined by the type of J_{ij} ensemble,

$$J_{\alpha\beta} = J_0 r_\alpha r_\beta, \quad \sum_{\alpha=1}^{N_0} r_\alpha^2 = 1 \tag{9}$$

Here we must note that the assumption that only one of eigenvalues of $J_{\alpha\beta}$ is finite, while all others are of the order $1/N$, is rather rough. It results in merging of the condensation points of all modes except one and makes the transition to be more sharp, while actually some modes will condense somewhere between $T = J$ and $T = J - J_0$. Yet the approximation in Eq. 9 allows to obtain analytical results which agree qualitatively with experiments, so it could be a starting point for more precise theory, accounting for distribution of the condensation temperatures of fractal modes.

Further we will show that the form of r_α could be determined from the fact that matrix $J_{\alpha\beta}$ defines the type of the ensemble's ground state.

Then Eq. 5 becomes

$$F = -\frac{J}{2} \sum_{\alpha=1}^{N_0} N_\alpha l_\alpha^2 + \frac{J_0}{2} (\sum_{\alpha=1}^{N_0} \sqrt{N_\alpha} r_\alpha l_\alpha)^2 - T \sum_{\alpha=1}^{N_0} N_\alpha S_\alpha (l_\alpha, m_\alpha) - NHm \tag{10}$$

Partial entropies $S_\alpha (l_\alpha, m_\alpha)$ in Eq. 6 can be represented as

$$S_\alpha (l_\alpha, m_\alpha) = \ln 2 - \max_{\varphi,\psi} \left[\varphi m_\alpha + \psi l_\alpha - N_\alpha^{-1} \sum_i \theta (|e_i^\alpha|) \ln ch \left(\varphi + \psi e_i^\alpha \sqrt{N_\alpha} \right) \right]$$

The values $\varphi_\alpha, \psi_\alpha$ corresponding to the maximum are determined by the equations

$$m_\alpha = N_\alpha^{-1} \sum_i th \left(\varphi_\alpha + \psi_\alpha e_i^\alpha \sqrt{N_\alpha} \right), \quad l_\alpha = N_\alpha^{-1/2} \sum_i e_i^\alpha th \left(\varphi_\alpha + \psi_\alpha e_i^\alpha \sqrt{N_\alpha} \right) \tag{11}$$

Differentiating potential in Eq. 10 over l_α and m_α we get the equations of state

$$T\varphi_\alpha = H, \quad J_0 N_\alpha^{-1/2} r_\alpha \sum_\beta N_\beta^{1/2} r_\beta l_\beta - J l_\alpha + T\psi_\alpha = 0 \tag{12}$$

The stable solutions of Eqs. 11 - 12 corresponding to the minima of F must have the positively defined Hessian

$$G_{\alpha\beta} = \delta_{\alpha\beta} \left\{ T \left[1 - \sum_i (e_i^\alpha)^2 th^2 \left(\varphi_\alpha + \psi_\alpha e_i^\alpha \sqrt{N_\alpha} \right) \right]^{-1} - J \right\} + J_0 r_\alpha r_\beta \tag{13}$$

For $H = 0$, $T = 0$ we have from Eqs. 11 - 12

$$m_\alpha = N_\alpha^{-1} \sum_i sign \left(e_i^\alpha l_\alpha \right)$$

$$|l_\alpha| = N_\alpha^{-1/2} \sum_i |e_i^\alpha|$$

Thus there are 2^{N_0} stable solutions of Eqs. 11 - 12 in this case which differ by the l_α signs . If the ensemble of random J_{ij} has ground states with $m = 0$ in almost all realizations then we may provide the minimal energy for the states with $m = 0$ putting $\sum_{\alpha=1}^{N_0} \sqrt{N_\alpha} r_\alpha l_\alpha = cm$. There is the unique function $r_\alpha(e_i^\alpha)$ obeying this condition for arbitrary l_α signs,

$$r_\alpha = \frac{c' \sum_i sign(e_i^\alpha)}{\sum_i |e_i^\alpha|} \tag{14}$$

Here c' is a normalization constant.

Further we consider only disordered magnets with zero magnetization in ground states such as dilute antiferromagnets, spin glasses and dilute ferromagnets with dipole interaction [25]. According to the above considerations on the form of e_i^α in dilute magnets and binary spin glasses(see Eq. 7), Eq. 14 can be represented as

$$r_\alpha = u_{1\alpha}(N_\alpha / \sum_\beta N_\beta u_{1\beta}^2)^{1/2} \tag{15}$$

Thus, in the approach outlined, the study of metastable states in the mentioned above magnets consists in the finding of stable solutions of Eqs. 11 - 12, with r_α in the form of Eq. 15. Spin configurations corresponding to the obtained l_α, m_α can be found from the expression

$$\langle S_i \rangle_T = \sum_\alpha N_\alpha^{-1/2} e_i^\alpha \frac{l_\alpha - u_{1\alpha} m_\alpha}{1 - u_{1\alpha}^2} + \sum_\alpha N_\alpha^{-1} \theta \left(|e_i^\alpha| \right) \frac{m_\alpha - u_{1\alpha} l_\alpha}{1 - u_{1\alpha}^2} \tag{16}$$

The averaging over disorder reduces to the averaging of solutions over random e_i^α and $J_0 << J$. Note that the localization threshold J is not random quantity being the characteristics of the whole ensemble of random J_{ij}.

Smallness of $J_0 > 0$ means that corresponding distribution function must have sufficiently narrow bounded support, i. e. the possible J_0 values must be smaller than some $\overline{J} > 0$ obeying the condition $\overline{J} << J$. Contrary to the case of the sums of macroscopic numbers of variables $u_{n\alpha}$ in Eq. 8, there are no reason to suppose the fluctuations of J_0 to be self-averaging, that is, that $\langle J_0^k \rangle \to \langle J_0 \rangle^k$ when $N \to \infty$. So the thermodynamic parameters of metastable states in the inhomogeneous magnetic phases will not be the self-averaging quantities being determined by the different J_0 values in different samples. Note that the absence of the self-averaging property of the stable thermodynamic parameters has been also observed in the numerical studies of disordered magnets [1, 26].

3   Thermodynamics near transition

In the absence of external field, the equations of state, Eqs. 11 - 12, have unique paramagnetic solution at $T > J$ and a number of stable solutions appears at $T < J$. Thus at $T = J$, $H = 0$ a transition from paramagnetic phase into inhomogeneous magnetic one takes place. Let us consider thermodynamics in the vicinity of this transition which is defined by the condition

$$l_\alpha, m_\alpha << 1 \tag{17}$$

In this case the equations for magnetizations of condensing modes, m_α, follow from Eqs. 11 - 12,15

$$\tau m_\alpha + \tau_0 u_{1\alpha}^2 mN/\sum_\beta N_\beta u_{1\beta}^2 + u_{4\alpha}m_\alpha^3/3u_{1\alpha}^2 = u_{1\alpha}^2 H/J \tag{18}$$

and l_α can be expressed via m_α,

$$u_{1\alpha}l_\alpha = m_\alpha + (u_{1\alpha}^2 - 1)H/J + (u_{3\alpha} - u_{1\alpha}u_{4\alpha}) m_\alpha^3/3u_{1\alpha}^3 \tag{19}$$

Here $\tau = 1 - J/T$, $\tau_0 = J_0/T$. Hessian, Eq. 13, has the form

$$T^{-1}G_{\alpha\beta} = \left(\tau + u_{4\alpha}m_\alpha^2/u_{1\alpha}^2\right)\delta_{\alpha\beta} + \tau_0 r_\alpha r_\beta$$

From Eqs. 17 - 18 it follows that $H \ll J$, $\tau \ll 1$, $\tau_0 \ll 1$.

It is natural to suppose the sums of macroscopic numbers of variables $u_{n\alpha}$ in Eq. 8 to be self-averaging quantities, so we can substitute them by their average values. Assuming that Eqs. 7 hold for the most disorder realizations, we get

$$\overline{u_{4\alpha}} = 1, \quad \overline{u_{3\alpha}} = \overline{u_{1\alpha}},$$

Let us also suppose that $\overline{u_{1\alpha}^2}$ do not depend on the α,

$$\overline{u_{1\alpha}^2} = \overline{u_1^2} \equiv N_0^{-1}\sum_{\alpha=1}^{N_0} u_{1\alpha}^2 \tag{20}$$

It seems that such approximation could not qualitatively change results. Giving it up one will just add some fluctuations to final expressions. Yet it allows to simplify essentially Eqs. 18 making possible their analytical study.

The constant $\overline{u_1^2}$ can be estimated using Eq. 7. From Eqs. 7,8 it follows

$$\overline{u_1^2} = N_0^{-1}\sum_\alpha \left(\nu_\alpha^+ - \nu_\alpha^-\right)^2$$

where ν_α^+ and ν_α^- are relative parts of positive and negative values of e_i^α, so $\overline{u_1^2} \leq 1$. Then $\overline{u_1^2} = 1$ in dilute dipole ferromagnets. In dilute antiferromagnet the difference $\nu_\alpha^+ - \nu_\alpha^-$ can be nonzero only due to uncompensated spins on the surface of d-dimensional antiferromagnetic clusters on which e_i^α are mostly localized. Hence $\nu_\alpha^+ - \nu_\alpha^-$ is of the order of the surface to volume ratio of d-dimensional antiferromagnetic clusters, so $\overline{u_1^2} \approx D^{-2}$, where D is the average diameter (in terms of lattice spacing) of these clusters. Evidently, D is a function of the concentration of antiferromagnetic atoms, which goes to infinity when concentration tends to 1.

In the binary spin glass, $\overline{u_1^2}$ depends on the concentration of ferromagnetic bonds p. In this case $\overline{u_1^2} = 1$ for $p > 1 - p_c$ and $\overline{u_1^2} = D^{-2}$ for $p < p_c$, p_c being the bond percolation threshold on the lattice of magnetic atoms. At $p_c < p < 1 - p_c$, $\overline{u_1^2}$ dependence on p can be qualitatively described as

$$\overline{u_1^2} = \frac{p - p_c + D^{-2}(1 - p - p_c)}{1 - 2p_c}$$

Here we also substitute N_α by their average value

$$N_\alpha = N/N_0.$$

Then, introducing instead of m_α reduced magnetizations μ_α ,

$$\mu_\alpha = m_\alpha/\sqrt{\overline{u_1^2}}$$

and dimensionless field

$$h = \sqrt{\overline{u_1^2}}H/T$$

Eqs. 20 can be represented as

$$\tau\mu_\alpha + \tau_0\mu + \mu_\alpha^3/3 = h \tag{21}$$

where $\mu = N_0^{-1}\sum_\alpha \mu_\alpha$.

At $\tau > 0$ Eq. 21 have unique paramagnetic solution with equal $\mu_\alpha = \mu$, which we denote as μ_0. It obeys the equation

$$(\tau + \tau_0)\mu_0 + \mu_0^3/3 = h \tag{22}$$

When $\tau < 0$ Eq. 21 have $2^{N_0} - 2$ stable inhomogeneous solutions besides μ_0 which can be represented as

$$\mu_\alpha = \sqrt{-\tau}\left(\sin\varphi + \sqrt{3}\sigma_\alpha\cos\varphi\right).$$

Here $\sigma_\alpha = \pm 1$ and $\varphi(\tau, \tau_0, h, \Delta)$ is the solution of the equation

$$3\tau_0\left(\sqrt{3}\Delta\cos\varphi + \sin\varphi\right) - 2\tau\sin 3\varphi = 3h\,|\tau|^{-1/2}, \tag{23}$$

$$\Delta = \sum_\alpha \sigma_\alpha/N_0 \tag{24}$$

The parameter Δ, varying in the interval $(-1, 1)$, defines the degree of the inhomogeneity of a metastable state and $\Delta = \pm 1$ correspond to the paramagnetic state with $\mu_\alpha = \mu$. All states with equal Δ have also the equal magnetizations

$$\mu = \sqrt{-\tau}\left(\sin\varphi + \sqrt{3}\Delta\cos\varphi\right), \tag{25}$$

and Edwards-Anderson order parameter

$$q = N^{-1}\sum_i \langle S_i\rangle_T^2 - m^2 \approx N_0^{-1}\sum_\alpha \mu_\alpha^2 - \overline{u_1^2}\mu^2 = 3\tau(\Delta^2 - 1)\cos^2\varphi + (1 - \overline{u_1^2})\mu^2 \tag{26}$$

as well as thermodynamic potential

$$4F/TN = (\tau_0 + \overline{u_1^2}\tau)\mu^2 + \tau q - 3h\mu - 4\ln 2 \tag{27}$$

These states are stable at $\tau + \mu_\alpha^2 > 0$, which is equivalent to the inequality

$$|\varphi| < \pi/6 \tag{28}$$

As $\tau_0 > 0$, the left side of Eq. 23 is a monotonously growing function of φ for $|\varphi| < \pi/6$. Hence, there is only one stable solution for φ at a given Δ, which exists in the interval of the fields

$$h_- < h < h_+, \quad h_\pm = \sqrt{3}h_{AT}\Delta/2 \pm h_c$$

$$h_{AT} = \sqrt{-3\tau\tau_0} \tag{29}$$

$$h_c = \sqrt{-\tau}(\tau_0/2 - 2\tau/3) \tag{30}$$

In this interval the solution of the Eq. 23 can be approximated by the quadratic function of the field

$$\varphi \approx \frac{\pi}{12h_c}\left[2h - \sqrt{3}h_{AT}\Delta - \frac{4(2 - \sqrt{3})\Delta h_{AT}(h_+ - h)(h - h_-)}{4h_c^2 - (2 - \sqrt{3})^2\Delta^2 h_{AT}^2}\right], \tag{31}$$

which gives the exact values $\varphi(h_\pm) = \pm\pi/6$ and $\varphi(\Delta h_{AT}) = 0$.

From the stability condition, Eq. 28, and Eqs. 23, 25 it follows that metastable states are stable in the region

$$9(\tau_0\mu - h)^2 < -4\tau^3$$

which is the band on the μ, h plane. The magnetization is a monotonously growing function of h and Δ inside this band so the field dependencies of magnetization can be represented as a set of uncrossed lines bounded from above and below by the $\mu_0(h)$ curve as shown in Fig. 1.

Apparently, the region on this figure, where metastable states exist, defines the form of hysteresis loop, which appears as a reaction on a slow AC field with amplitude greater than
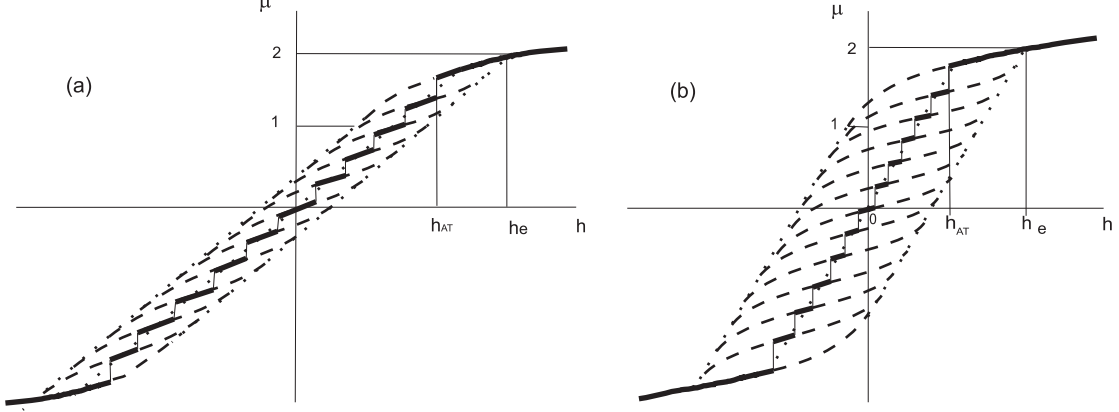
$$h_e = h_c + \sqrt{3}h_{AT}/2 \tag{32}$$

Figure 1: Field dependencies of magnetization in metastable states (dashed lines) and stable states (solid lines) near transition, (a) - $0 < -2\tau < 3\tau_0$, (b) - $3\tau_0 < -2\tau$.

There are certain temperature variations in the loop form as at $-2\tau < 3\tau_0$ only part of metastable states are stable at $h = 0$ and loop is rather slim, see Fig. 1(a), while at $-2\tau > 3\tau_0$ all metastable states are stable at $h = 0$ and loop became more thick, Fig. 1(b). Note also that when amplitude of AC field is less than h_e the form of hysteresis loop is defined by the field dependencies of magnetizations in corresponding metastable states.

Let us also present the expressions for (dimensionless) magnetic susceptibility $\chi = \partial\mu/\partial h$,

$$\chi^{-1} = \tau_0 - 2\tau\cos 3\varphi/(\cos\varphi - \sqrt{3}\Delta\sin\varphi),$$

entropy S and heat capacity C,

$$S = \ln 2 - (q + \overline{u_1^2}\mu^2)/2,$$

$$C = 1 + \chi\left[\frac{3}{2}\cdot\frac{\tau_0\left(1 - \Delta^2\right)}{1 - \sqrt{3}\Delta tg\varphi} - \tau_0 - \tau\right]$$

The above expressions allow to get some notion about the field and temperature dependencies of these quantities. Thus at the boundaries of stability region, $h = h_\pm$, q and χ^{-1} has the lowest values

$$q = 9\tau\left(\Delta^2 - 1\right)/4 - \left(1 - \overline{u_1^2}\right)\tau\left(3\Delta \pm 1\right)^2/4, \quad \chi = 1/\tau_0, \tag{33}$$

while magnetization, entropy and heat capacity are

$$\mu = \sqrt{-\tau}\left(3\Delta \pm 1\right)/2, \quad S = \ln 2 + \tau\left(5 \pm 3\Delta\right)/4, \quad C = 3\left(1 \pm \Delta\right)/2 - \tau/\tau_0 \tag{34}$$

When $|h|$ goes to h_e, Eq. 32, the more homogeneous states with $\Delta \to \pm 1$ stay stable and their magnetization tends to $\mu_0\left(\pm h_e\right) = \pm 2\sqrt{-\tau}$. However the limiting values of magnetic susceptibility and heat capacity differ from those in paramagnetic state:

$$\chi_0^{-1} = \tau + \tau_0 + \mu_0^2, \quad C_0 = \mu_0^2/\left(\tau + \tau_0 + \mu_0^2\right).$$

In the middle of the stability band (at $\varphi = 0$ or $h = \Delta h_{AT}$) we get:

$$\mu = \Delta\sqrt{-3\tau}, \quad q = -3\tau\left(1 - \overline{u_1^2}\Delta^2\right), \quad \chi^{-1} = \tau_0 - 2\tau, \quad S = \ln 2 + 3\tau/2, \quad C = \frac{3}{2}\left(1 - \Delta^2\frac{\tau_0}{\tau_0 - 2\tau}\right).$$

In this case with the diminishing of inhomogeneity when $\Delta \to \pm 1$ or $h \to \pm h_{AT}$, μ, χ, S and C tend to the corresponding values of the paramagnetic phase.

The Almeida-Thouless field h_{AT}, Eq. 29, determines (to the order N_0^{-1}) the point of the transition into the paramagnetic phase. To show this let us find the values Δ_{eq} corresponding to the states with the lowest potential. Differentiating F, Eq. 27, over Δ and using Eqs. 23, 25 and Eq. 26 we get

$$\frac{\partial F}{\partial\Delta} = -NT\sin\varphi\cos^3\varphi, \quad \frac{\partial^2 F}{\partial\Delta^2}|_{\varphi=0} > 0$$

Thus the lowest value of potential have the states with $\varphi = 0$ at given τ and h. One can see that Eq. 23 has solution with $\varphi = 0$ when $\Delta = h/h_{AT}$ which is possible at $h^2 < h_{AT}^2$. When $h^2 > h_{AT}^2$ $F(\Delta)$ has no minima inside the region $\Delta^2 < 1$ in which it is defined and the minimal values occur at its boundaries for $\Delta_{eq} = sign(h)$. So the transition into paramagnetic state takes place at $h = \pm h_{AT}$.

As Δ is a rational number of the form $2n/N_0 - 1$ (cf. Eq. 24) it can not be exactly equal to h/h_{AT} at all $h^2 < h_{AT}^2$. Hence Δ_{eq} is defined so that $|\Delta - h/h_{AT}|$ is minimal and can be represented as

$$\Delta_{eq} = \sum_{n=1}^{N_0 - 1} \left(\frac{2n}{N_0} - 1 \right) \theta \left(N_0^{-2} - \varepsilon_n^2 \right) + sign(h)\theta \left[h^2 - \left(\frac{N_0 - 1}{N_0} \right) h_{AT}^2 \right]$$

$$\varepsilon_n \equiv \frac{h}{h_{AT}} - \frac{2n}{N_0} + 1$$

Thus at $h^2 < h_{AT}^2$ series of transitions between inhomogeneous magnetic states takes place at fields

$$h_n = h_{AT} \left(\frac{2n + 1}{N_0} - 1 \right)$$

The value of φ_{eq} at $h^2 < h_{AT}^2$ corresponding to Δ_{eq} is, see Eq. 23,:

$$\varphi_{eq} = \frac{\sqrt{3}\tau_0}{\tau_0 - 2\tau} \sum_{n=1}^{N_0 - 1} \varepsilon_n \theta \left(N_0^{-2} - \varepsilon_n^2 \right)$$

Inserting Δ_{eq} and φ_{eq} in the Eqs. 25, 26 we get the equilibrium values μ_{eq} and q_{eq} for $N \to \infty$:

$$\mu_{eq} = \frac{h}{\tau_0} \theta \left(h_{AT}^2 - h^2 \right) + \mu_0 \theta \left(h^2 - h_{AT}^2 \right)$$

$$q_{eq} = -3\tau \left(1 - h^2/h_{AT}^2 \right) \theta \left(h_{AT}^2 - h^2 \right) + \left(1 - \overline{u_1^2} \right) \mu_{eq}^2$$

Differentiating m_{eq} over h we get the equilibrium susceptibility

$$\chi_{eq} = \tau_0^{-1} \theta \left(h_{AT}^2 - h^2 \right) + \left(\tau + \tau_0 + \mu_0^2 \right)^{-1} \theta \left(h^2 - h_{AT}^2 \right).$$

The equilibrium entropy can be obtained by the differentiation of the equilibrium potential which to the ε_n^2 order is

$$F_{eq} = F \left(\Delta = h/h_{AT} \right) - T S_{conf}$$

where configurational entropy S_{conf} is determined by the logarithm of the number of states with the same potential F,

$$S_{conf} = N^{-1} \ln \binom{N_0}{N_0 \left(1 - \Delta_{eq} \right)/2}$$

S_{conf} is of the order N_0/N and can be neglected. Hence at $N_0 \to \infty$

$$S_{eq} = \ln 2 + \frac{3\tau}{2} \theta \left(h_{AT}^2 - h^2 \right) - \frac{\mu_0^2}{2} \theta \left(h^2 - h_{AT}^2 \right)$$

For the equilibrium heat capacity we get

$$C_{eq} = \frac{3}{2} \theta \left(h_{AT}^2 - h^2 \right) - \frac{\mu_0^2}{\tau + \tau_0 + \mu_0^2} \theta \left(h^2 - h_{AT}^2 \right).$$

Let us note that μ_{eq}, q_{eq} and S_{eq} are continuous at $h^2 = h_{AT}^2$, while χ_{eq} and C_{eq} undergoes jumps at the transition into paramagnetic phase.

We must also note that the average equilibrium parameters are generally unobservable quantities due to the macroscopic free energy barriers between metastable states. Probably, the experimental values, which are rather close to them, are obtained after cooling in small external fields down to T just below J (field-cooled (FC) regime) [1, 3] when barriers between metastable states are relatively small and system could relax into the lowest (or close to it) state at a sufficiently slow cooling. In zero field cooled (ZFC) regime when field is applied after cooling below $T = J$ in zero field, the observed thermodynamic parameters would differ from equilibrium ones as the system would at first be trapped in the state with $\Delta = 0$ and will stay in it if applied field does not exceed h_c, cf. Eq. 30 and Fig.1. Thus at $h < h_c$ the
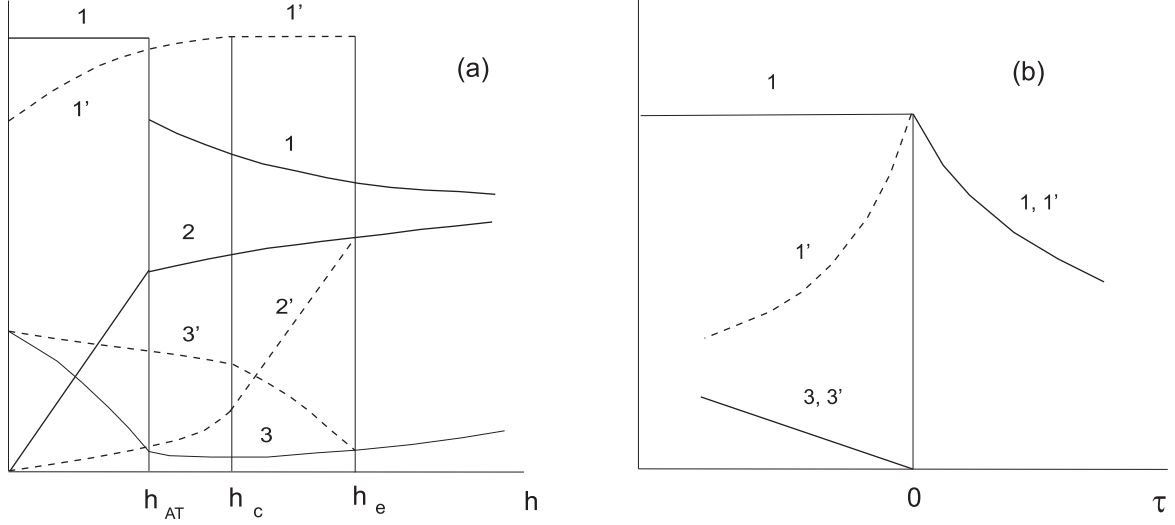
Figure 2: Field (a) and temperature (b) dependencies of some thermodynamic parameters in FC and ZFC regimes, 1-FC susceptibility, 1'-ZFC susceptibility, 2-FC magnetization, 2'-ZFC magnetization, 3-FC EA order parameter, 3'-ZFC EA order parameter

ZFC parameters are those of $\Delta = 0$ metastable states. Their values can be obtained from the general expressions at $\Delta = 0$ and $\varphi = \pi h/6h_c$, see Eq. 31. When applied field $h > h_c$, the system relaxes into the metastable state at the boundary of stability region (on the lower branch of hysteresis loop) with

$$\Delta_{ZFC} = 2(h - h_c)/\sqrt{3}h_{AT}.$$

Inserting this Δ_{ZFC} in Eq. 33 and Eq. 34 (with plus sign) we get the values of thermodynamic parameters the observed quantities would relax to in ZFC regime at $h_c < h < h_e$. And when $h > h_e$ ZFC parameters correspond to those of paramagnetic state. The field and temperature dependencies of some thermodynamic parameters in FC and ZFC regimes are shown in Fig. 2.

Similarly, the regions of existence of metastable states, see Fig. 1, and their parameters define the other quantities which are determined in the slow nonequilibrium processes, such as thermo-remanent magnetization, μ_{TRM}, which remains after FC process and subsequent switching off the field, and isothermal remanent magnetization, μ_{IRM}, remaining after ZFC process followed by the application for some time (longer than the intravalley relaxation time) an external field [1, 3]. Apparently, μ_{IRM} is nonzero only at $h > h_c$ and an expression for it can be obtained from Eq. 25 at

$$\Delta_{IRM} = \min\left[1, 2h_c/\sqrt{3}h_{AT}, 2(h - h_c)/\sqrt{3}h_{AT}\right]$$

$$\varphi_{IRM} \equiv \varphi(h = 0, \Delta_{IRM}) \approx -\frac{\sqrt{3}\pi h_{AT}\Delta_{IRM}}{12h_c}$$

μ_{IRM} can also be obtained from Eq. 25 with

$$\Delta_{TRM} = \min\left(1, 2h_c/\sqrt{3}h_{AT}, h/h_{AT}\right)$$

$$\varphi_{TRM} \equiv \varphi(h = 0, \Delta_{TRM}) \approx -\frac{\sqrt{3}\pi h_{AT}\Delta_{TRM}}{12h_c}$$

The field dependencies of μ_{TRM} and μ_{IRM} are shown in Fig. 3(a). At $h > h_{TRM} = min(2h_c/\sqrt{3}, h_{AT})$ μ_{TRM} becomes constant, while above $h_{IRM} = min(2h_c, h_e)$ μ_{IRM} also saturates at the same value. This value of saturation magnetization is

$$\mu_\infty = \theta(3\tau_0 + 2\tau)2(-\tau)^{3/2}/3\tau_0 + [-3(\tau_0 + \tau)]^{1/2}\theta(-3\tau_0 - 2\tau)$$

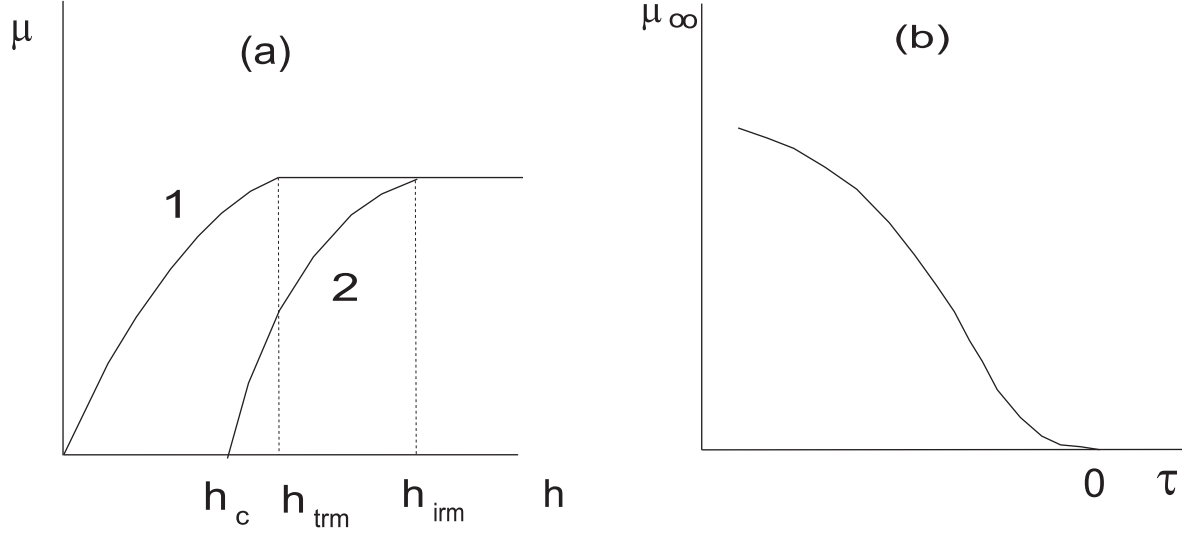Temperature dependence of μ_∞ is shown in Fig. 3(b)

Figure 3: (a) - field dependencies of μ_{TRM} (1) and μ_{IRM} (2), (b) - temperature dependence of saturation magnetization μ_∞

4 Conclusions

The main result of the present work consists in the qualitative, but complete description of properties of all metastable states in the inergodic phases of random Ising magnets with zero ground state magnetization and elucidation of their relations to the parameters of slow irreversible processes. The results depicted in Fig. 1 allow to describe every conceivable irreversible process with arbitrary sequence of field and temperature changes. Qualitative agreement of the obtained here parameters of some such processes with experiments and numerical studies [1]- [4] justifies the approximations used in Eqs. 7, 9, 15, 20 and shows that the condensation of macroscopic number of sparse fractal modes near the localization threshold do can be a possible mechanism of the appearance of exponentially large number of metastable states in disordered Ising magnets. The argument in favor of this mechanism is also the structure of the set of these states related via overturns of independent spin groups, corresponding to fractal modes, as just the same relations between ground states in $3d$ Ising spin glass with $\pm J$ exchange are revealed in recent numeric studies [8].

Let us also note that obtained here results are expressed solely in terms of statistical characteristics of random exchange matrix. So the present approach could serve as a starting point for the developing of more precise quantitative theory of metastable states in disordered magnets. Such theory should be based on the detailed studies of the properties of random J_{ij} eigenvectors near the localization threshold, which we were compelled to describe here in terms of phenomenological suppositions. One of the tasks of this theory could be, in particular, the test of the universality of the properties of the magnets with zero ground state magnetization, as the results for them differ in present approach just by the values of $\overline{u_1^2}$.

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